UNIVERSITY OF LONDON

GENERAL CERTIFICATE OF EDUCATION EXAMINATION

SUMMER 1970

Advanced Level

MATHEMATICS 1

PURE MATHEMATICS

Three hours

Answer EIGHT questions.

1. (i) If the roots of the quadratic equation $x^2 + bx + c = 0$ are α and β, find the quadratic equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$.

(ii) Find the possible values of x if

$$(2)^{2x+1} = 3\,(2)^x - 1$$

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Turn over

2. (i) (*a*) If $z = 3 + 4i$, express $z + \dfrac{25}{z}$ in its simplest form.

(*b*) If $z = x + yi$, find the real part and the imaginary part of $z + \dfrac{1}{z}$.

Find the locus of points in the Argand diagram for which the imaginary part of $z + \dfrac{1}{z}$ is zero.

(ii) State de Moivre's theorem for a positive integral index, and use it to show that

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

where $t = \tan\theta$.

3. (i) Find the number of ways in which a committee of 4 can be chosen from 6 boys and 6 girls

(*a*) if it must contain 2 boys and 2 girls,

(*b*) if it must contain at least 1 boy and 1 girl,

(*c*) if either the oldest boy or the oldest girl must be included, but not both.

(ii) If n is an integer, use the *method of induction* to prove that

$$1^3 + 2^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

4. Expand $\sqrt{(1 - 4x)}$ and $1 - 2x(1 - px)^q$ in ascending powers of x. If these expansions are identical up to the term in x^3, find p and q. State for these values of p and q the common set of values of x for which both expansions are valid.

By putting $x = \frac{1}{6}$, show that the cube root of 2 is approximately $3 - \sqrt{3}$ and by putting $x = \frac{1}{27}$ obtain an approximation for the cube root of 3.

5. A circle with centre P and radius r touches externally both the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x + 8 = 0$. Prove that the x-coordinate of P is $\frac{1}{3}r + 2$, and that P lies on the curve

$$y^2 = 8(x - 1)(x - 2).$$

6. A variable line $y = mx + c$ cuts the fixed parabola $y^2 = 4ax$ in two points P and Q. Show that the coordinates of M, the mid-point of PQ, are

$$\left(\frac{2a - mc}{m^2}, \frac{2a}{m}\right).$$

Find one equation satisfied by the coordinates of M in each of the following cases:

(a) if the line has fixed gradient m,

(b) if the line passes through the fixed point $(0, -a)$.

7. (i) Find, in radians, the general solution of the equation

$$4 \sin \theta = \sec \theta .$$

(ii) If $\sin \theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$, show that θ is either a multiple of $\frac{1}{2}\pi$ or a multiple of $\frac{2}{5}\pi$.

8. (i) If y is inversely proportional to $(x + 1)$, show that

(a) $(x + 1)\dfrac{dy}{dx} + y = 0$,

(b) $y\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2$.

(ii) BC is a chord of a circle with centre O and radius a. The mid-point of BC is M, and MO is extended to A, where $OA = 2a$. Find, to the nearest minute, the angle BOM so that the triangle ABC has maximum area.

9. (i) If $a \neq 0$ and $\int_0^a x(x-1)\,dx = 0$, find a and evaluate

$$\int_0^a x^2(x-1)^2\,dx .$$

(ii) Prove that $\int_{\pi/6}^{\pi/3} \sin 4x \sin x\,dx = (3\sqrt{3} - 7)/60$.

10. (i) Find the area of the finite figure bounded by the curve $y = x(2 - x)$ and the line $y = kx$, where $0 < k < 2$.

(ii) The finite area in the first quadrant bounded by the curves $y = 4/x$, $y = 4/x^2$, and the line $x = 2$ is rotated once about the x-axis. Find the volume of the solid of revolution described.