UNIVERSITY OF LONDON

GENERAL CERTIFICATE OF EDUCATION EXAMINATION

SUMMER 1970

Advanced Level

MATHEMATICS 2

PURE MATHEMATICS

Three hours

Answer EIGHT questions.

1. (i) Find the set of real values of x for which the series

$$\sum_{r=1}^{\infty} (-1)^r \left(\frac{1+x}{1+x^2} \right)^r$$

is convergent.

Find the value of x for which the sum to infinity of this series is a maximum.

(ii) Find the sum of the series $\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$.

2. (i) Form the cubic equation whose roots are the squares of the roots of the equation $x^3 + ax^2 + bx + c = 0$.

(ii) When the polynomial $x^3 + ax^2 + bx + c$ is divided by $(x-1)$, $(x-2)$, $(x-3)$ the remainders are 24, 60, 120 respectively. Solve the equation $x^3 + ax^2 + bx + c = 0$.

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Turn Over

3. (i) Express ln $(1 - x)$ and ln $(2 + x)$ as series of ascending powers of x up to and including the terms in x^3. Show that

$$\ln (2 - x - x^2)$$
$$= \ln 2 - \frac{x}{2} - \frac{5x^2}{8} - \frac{7x^3}{24} - \dots - \frac{x^n}{n}\left[1 + \left(-\frac{1}{2}\right)^n\right] - \dots$$

and state the range of values of x for which this infinite series converges.

(ii) Show that the nth term of the series

$$\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \dots + \frac{2n}{(2n+1)!} + \dots$$

can be expressed in the form $\frac{1}{(2n)!} - \frac{1}{(2n+1)!}$, and hence show that the sum of the infinite series is $1/e$.

4. (i) If $z = \cos\theta + i\sin\theta$, show that

$$z^n + z^{-n} = 2\cos n\theta \text{ and } z^n - z^{-n} = 2i\sin n\theta .$$

Hence deduce that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

(ii) Given that $1, \omega, \omega^2$ are the cube roots of unity, prove that

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$$
$$= a^3 + b^3 + c^3 - 3abc .$$

5. A tetrahedron has one vertex at the origin O and the other three vertices at the points $A(0, 2, 2)$, $B(1, 2, 3)$, $C(3, 1, 6)$. Find

(a) the equation of the plane ABC,

(b) the coordinates of the foot of the perpendicular from O to the plane ABC,

(c) the volume of the tetrahedron.

6. Show that the equation of the normal at $P(ct\,,\ c/t)$ to the rectangular hyperbola $xy = c^2$ is $t^3x - ty = c(t^4 - 1)$.

The tangent at P meets the x-axis at A and the y-axis at B and the normal at P meets the y-axis at C. If M is the mid-point of AC find the equation of the locus of M. Show that the area of the triangle ABC is four times the area of the triangle AMP.

7. The tangent at $P\,(a\cos\theta\,,\ b\sin\theta)$ to the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

cuts the y-axis at Q. The normal at P is parallel to the line joining Q to one focus S'. If S is the other focus, show that PS is parallel to the y-axis.

8. (i) If $y = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$, find $\dfrac{dy}{dx}$ in its simplest form.

(ii) If $s = ae^{-bt}\sin ct$, where $a\,,\ b\,,\ c$ are constants, show that

$$\frac{d^2s}{dt^2} + 2b\frac{ds}{dt} + (b^2 + c^2)\,s = 0\,.$$

(iii) Two particles P and Q are at rest, each at a distance k metres from the origin, P being on the x-axis and Q on the y-axis. At the same instant each starts moving towards the origin, P with constant speed k m/s and Q with constant acceleration k m/s². Find the rate at which the distance between P and Q is decreasing half a second later.

9. (i) Find (*a*) $\displaystyle\int x^2\sin x\,dx$, (*b*) $\displaystyle\int \frac{dx}{x^2\sqrt{(9+x^2)}}$.

(ii) The tangent at any point P on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, meets the axes at A and B. The normal at P meets the axes at C and D. Show that AB is constant in length and that when $\theta = \frac{1}{8}\pi$, $CD = 2AB$.

10. Sketch the curve $y = \dfrac{x^2+1}{x^2-1}$.

Show that the x-coordinate of the centroid of the area bounded by the curve, the x-axis and the lines $x = 2$ and $x = 3$, is approximately 2·48.